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Heuristic approach to the strong-coupling regime of the Kardar-Parisi-Zhang equation

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We propose a heuristic approach of treating the strong-coupling regime of the Kardar-Parisi-Zhang (KPZ) equation. The method, which suggests that $d_{uc}=4$ is the upper critical dimension of the KPZ equation, enables one to use the ε expansion below $d=4$ substrate dimensions to compute the critical exponents in the strong-coupling regime. We compute the dynamic exponent z and the roughness exponent χ to the first order in $\varepsilon=4-d$ as $z=2-(4-d)/5$ and $\chi=(4-d)/5$. [S1063-651X(97)50705-7]

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The nonequilibrium dynamics of surfaces continues to attract large interest over the recent years (for reviews see [1–4]). These systems exhibit critical properties similar to those of equilibrium critical phenomena. A phenomenological equation describing the dynamics of surfaces is the Kardar-Parisi-Zhang (KPZ) equation [5]

$$\frac{\partial h}{\partial t}=\nu_0\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\mathbf{x},t), \tag{1}$$

where $h(\mathbf{x},t)$ is a single-valued function, which describes the height profile above a basal d-dimensional substrate $\mathbf{x}$ in the comoving coordinate system, λ is responsible for the lateral growth, ν_0 is the surface tension, and the noise $\eta(\mathbf{x},t)$ has a Gaussian distribution with $\langle\eta(\mathbf{x},t)\rangle=0$, and

$$\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle=2D_0\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{2}$$

Equation (1) is now widely accepted to describe growth processes such as Eden model, growth by ballistic deposition, etc. The KPZ equation is also related to randomly stirred fluids (Burgers equation [6]), dissipative transport in the driven-diffusion equation [7], the directed polymer problem in disordered media [8], and the behavior of flux lines in superconductors [9].

The height-height correlation function has for $d\leqslant 2$ the following scaling form:

$$\langle(h(\mathbf{x},t)-h(\mathbf{x}',t'))^2\rangle=|\mathbf{x}-\mathbf{x}'|^{2\chi}f(|t-t'|/|\mathbf{x}-\mathbf{x}'|^z), \tag{3}$$

where χ and z are the roughness and the dynamic exponents, respectively. In the rough phase the exponents obey the scaling relation $\chi+z=2$, which follows from the invariance of Eq. (1) with respect to an infinitesimal tilting of the surface $h\rightarrow h+\epsilon\mathbf{x}$, $\mathbf{x}\rightarrow\mathbf{x}+\lambda\epsilon t$ [5]. For $d>2$ there are two distinct regimes. For $\lambda<\lambda_c$, the interface is smooth, while for $\lambda>\lambda_c$ it is rough and is expected to obey the scaling law (3) with the nontrivial roughness exponent χ. The scaling behavior for both equilibrium and dynamical phenomena far from equilibrium is usually described by using the renormalization group (RG) method. However, RG method fails to describe the strong-coupling regime of the KPZ equation. The RG analysis of Eq. (1) up to the one-loop order in the vicinity of $d=2$ yields the effective coupling constant, which increases under renormalization and seems to make the perturbation approach useless. The strong-coupling regime is so far analytically poorly understood (for recent studies see [10–20]). References [12,13,18] employ mode-coupling techniques, yield numerical estimates of the dynamical exponents and suggest d_{uc} slightly less than four or equals four [18]. References [20] and [19] (see also [4], Sec.6.4.IV) suggest that $d_{uc}=4$.

In the present paper we will present a heuristic approach of treating the strong-coupling regime of the KPZ equation and will compute the critical exponents z and χ below four dimensions by using the ε expansion to first order in $\varepsilon=4-d$. The basic idea of this approach is based on the suggestion that the increase of the effective coupling constant of Eq. (1), $g\simeq D\lambda^2\nu_0^{-3}$, under renormalization in the vicinity of $d=2$ is due to the fact that for $d>2$ the critical value of λ, λ_c, becomes a relevant quantity. The necessity of generation of λ_c in treating Eq. (1) is the reason of the failure of the standard RG technique in handling the strong-coupling regime of the KPZ equation. For $d\geqslant 2$ the perturbation expansions explicitly depend on the ultraviolet cutoff l_0. We suggest that this dependence on l_0, which is a local quantity, is responsible for the appearance of the threshold value λ_c, which is also expected to be a local quantity. In the light of these ideas it is tempting to reorganize the perturbation expansions so that the cutoff dependence of the perturbation expansions appearing for $d\geqslant 2$ will be converted to a threshold of the coupling constant [21]. The approach proposed consists of two steps. The first step consists in carrying out the renormalization of the perturbation expansions in the vicinity of $d=2$. Using a special matching condition, which is a heuristic one and is introduced by hand, enables one to reorganize the perturbation expansions, so that the pole of the effective coupling constant disappears. Instead to diverge at a finite length, the effective coupling constant behaves as the square of an infrared length (for $d>2$, when $\lambda>\lambda_c$). This behavior of the effective coupling constant results, first, in shifting the critical dimension from being initially $d=2$ to $d=4$. Second, the effective coupling constant yields the desired threshold of λ for $d>2$ and the expansion parameter in the rough phase becomes proportional to l^{4-d}, with l being an infrared length. The second step consists in performing the RG analysis of these perturbation series in the vicinity of $d=4$. The possibility for carrying out the ε expansion in $d<4$ substrate dimensions is due to the fact that (i) the coupling constant does depend on the length as l^{4-d}, and (ii) the regularized parts of perturbation expansions contain $1/(4-d)$ poles. The importance of $1/(4-d)$ poles for the strong-coupling regime was previously emphasized in [16] and [17]. The study of the singularities at $4-d$ and their consequences for the roughening transition has been treated in [23] within the picture of directed polymer (DP) by using the replica method.

The renormalization prescriptions of the parameters entering Eq. (1) can be obtained in a standard manner [6,8,11] and are

$$\nu=\nu_0\left(1+\frac{2-d}{4d}\,\frac{\Gamma(2-d/2)}{2-d}g_0 l^{2-d}+\cdots\right), \tag{4}$$

$$D=D_0\left(1+\frac{1}{4}\,\frac{1}{2-d}g_0 l^{2-d} + \frac{1}{8}\,\frac{\Gamma(2-d/2)-1}{1-d/2}g_0 l^{2-d}+\cdots\right), \tag{5}$$

$$\lambda'=\lambda, \tag{6}$$

where $g_0=[2/(4\pi)^{d/2}]D_0\lambda^2/\nu_0^3$. The length l in Eqs. (4) and (5) plays the role of a cutoff at the lower limit of the integral, $\int_k 1/(k^2+l^{-2})$, appearing in the one-loop correction to ν and D. Depending on the conditions l can be identified with $(\nu_0 t)^{1/2}$, the inverse external momentum at which the renormalization is performed, the size of the system, etc.

In carrying out the renormalization in the vicinity of $d=2$ only the second term on the right-hand side of Eq. (5) has to be taken into account. The result of the integration of the RG equation for D from the cutoff l_0 to the cutoff l_m is

$$D=D_0\{1-[1/(2-d)]g_0\beta'(l_m^{2-d}-l_0^{2-d})\}^{-\gamma_D'/\beta'}, \tag{7}$$

where $\beta'=1/4$ and $\gamma_D'=1/4$. We note that the effective coupling constant g behaves in the same way as D. The ultraviolet cutoff l_0 introduced in Eq. (7) enables one to extend Eq. (7) for $d\geqslant 2$.

The crucial point of our approach to treat the strong-coupling regime consists in the use of the following matching condition for the length l_m in Eq. (7):

$$l_m^{-2}=l^{-2}+l_c^{-2}, \tag{8}$$

where l is the relevant infrared length of the problem. For a DP in the presence of an extended defect Eq. (8) can be legitimated in a rigorous way [24]. It is known that in order to study the critical phenomena by using the RG method it is necessary to use a condition (matching condition) giving a relation between the final length l_m appearing in the effective parameters of UV regularized theory and the relevant infrared length of the problem. The information over the infrared behavior is put always into the matching condition by hand. In this sense (8) is a generalized matching condition. The form of Eq. (8) is dictated by demanding that the time has to be involved in Eq. (8) as $1/t$, due to the fact that at the initial stage of the growth l^2 is proportional to $\nu_0 t$. This circumstance selects Eq. (8) among the more general matching condition, $l_m^{-n}=l^{-n}+l_c^{-2}$, $n=2,\,3,\,\ldots$, which, of course, for $n\neq 2$ would lead to different conclusions. However, $n\neq 2$ is inconsistent with $1/(4-d)$ poles in Eqs. (4) and (5) as it will be discussed below. The quantity l_c^{-1} in Eq. (8) can be identified with an external momentum, the inverse size of the system, etc. Equation (8) does not contradict the perturbation theory on small scales ($l^{-1}\gg l_c^{-1}$). When $l_c^{-1}\neq 0$, l will become in general irrelevant for $l\to\infty$. It turns out that a special choice of l_c in Eqs. (7) and (8) gives the expected threshold for the coupling constant for $d>2$ and has the consequence that the length l remains relevant for $l\to\infty$. Inserting Eq. (8) into Eq. (7) and demanding that the denominator in Eq. (7) behaves linear in l^{-2} for small l^{-2} results in

$$D=2D_0/(g_0\beta' l_c^{4-d})l^2, \tag{9}$$

where in deriving Eq. (9) we have taken into account that $\gamma_D'/\beta'=1$. The crossover length l_c is obtained in different dimensions as

$$l_c=\begin{bmatrix} [(2-d)/(\beta' g_0)]^{1/(2-d)}, & d<2 \\ l_0\exp[1/(\beta' g_0)], & d=2 \\ \{\beta'/[(d-2)(g_c^{-1}-g_0^{-1})]\}^{1/(d-2)}, & d>2 \end{bmatrix}, \tag{10}$$

where $g_c=[(d-2)/\beta']l_0^{d-2}$ is the threshold value of the coupling constant. It is supposed that in Eq. (10) for $d>2$ the condition $g_0>g_c$ is fulfilled.

Due to the use of the matching condition (8), the UV regularized perturbation expansions are reorganized in such a way that a threshold of the bare coupling constant g_0 controlling the behavior of the interface for $d>2$ explicitly enters the perturbation expansions. The threshold value of the bare coupling constant g_c is determined by the microscopic cutoff l_0. The effective coupling constant $g=\lambda^2 D/\nu_0^3$ behaves like D and scales with the infrared length as $g\sim D\sim l^2$. The quadratic dependence of the coupling constant on the length l has the consequence that the critical dimension is shifted to $d=4$. This follows from the consideration of the dimensionless coupling constant $u_0=(\lambda^2 D/\nu_0^3)l^{2-d}$, which results after using Eq. (9) in $u_0\simeq(l/l_c)^{4-d}$. For $g_0<g_c$ and $d>2$ there is no a solution for l_c obeying the condition that the denominator in Eq. (7) behaves linear in l^{-2} for small l^{-2}. This results in irrelevancy of the nonlinear term in Eq. (1) for $d>2$ in the smooth phase $g_0<g_c$. At the transition, $g_0=g_c$, the exponents are given by $z_c=2$ and $\chi_c=0$ [25–27,11,17]. This is due to the fact that at the transition the effective coupling constant $u_0\simeq(l/l_c)^{4-d}$ is zero, since the crossover length l_c is infinite. Slightly above the transition the crossover length l_c behaves for $2<d<4$ according to Eq. (10) as $l_c\sim|g_0-g_c|^{-1/(d-2)}$, which agrees with the behavior of the correlation length below the transition [25,11].

We note that for $d>2$ the crossover length l_c and consequently the effective coupling constant u_0 has a finite limit for $g_0\rightarrow\infty$. In terms of the DP picture in a disordered medium g_0 is inversely proportional to the temperature. The independence of l_c on g_0 for large g_0 means that the properties of the surface do not depend sensitively on temperature at low temperatures, the prediction which is quite natural to be expected [26].

We now will compute the critical exponents in the strong-coupling regime by using the RG method. From Eqs. (4) and (5) we see that the regular parts of one loop corrections to ν and D contain $1/(4-d)$ poles. Due to this and the circumstance that the coupling constant behaves as l^{4-d}, it is tempting to perform the renormalization of the perturbation expansions in the vicinity of $d=4$. Note the consistency of the shift of the critical dimension due to Eq. (9) to $d_{uc}=4$, which is the consequence of the use of the matching condition (8), with the $1/(4-d)$ poles in the perturbation expansion. The procedure used depends on the assumption that the crossover to the strong-coupling regime is governed by the poles (4) and (5) evaluated at the Gaussian fixed point. A careful study of the shift $2-d\rightarrow 4-d$ and its consequence on the potential change of the prefactors in front of $1/(4-d)$ poles in Eqs. (4) and (5), which are responsible for the strong-coupling exponents, is necessary. The effective coupling constant $u_0\simeq(l/l_c)^{4-d}$ becomes now the bare coupling constant of the regularized perturbation expansions. According to Eqs.(4)–(6), the renormalization of u_0 is due to the renormalization of D and ν. The renormalized coupling constant u is expressed through D and ν in the same way as g through g_0, i.e., $u\simeq(D/D_0)/(\nu/\nu_0)^3)(l/l_c)^{4-d}$. The last term in Eq. (5) is negative, that means that D decreases under renormalization. Taking into account that ν increases under renormalization, we see that the dimensionless effective coupling constant u has a fixed-point value.

The differential equations of the RG for D and ν are obtained from Eqs. (4) and (5) as

$$l\partial\ln\nu/\partial l=\gamma_\nu u, \quad l\partial\ln D/\partial l=-\gamma_D u, \tag{11}$$

where $\gamma_\nu=1/8$ and $\gamma_D=1/4$ are computed from Eqs. (4) and (5) at $d=4$ by extrapolating the regular parts of these equations to $d\geqslant 2$. The flow equation for the dimensionless coupling constant u is obtained from Eqs. (11) as

$$l\partial u/\partial l=(4-d)u-(5/8)u^2. \tag{12}$$

The fixed-point value of u is obtained from Eq. (12) as $u^*=(8/5)(4-d)$. The quantities D and ν behave according to Eqs. (11) at the fixed point as

$$D\sim l^{-\gamma_D u^*}, \quad \nu\sim l^{\gamma_\nu u^*}. \tag{13}$$

The dynamic exponent z is defined by the relation $t\sim l^z$. The use of the relation $l^2=\nu t$ and the scaling law for ν given by the second relation in Eq. (13) yields $z=2-\gamma_\nu u^*$, which up to the first order in $4-d$ results in the following expression for the dynamic exponent:

$$z=2-(4-d)/5. \tag{14}$$

The roughness exponent can be obtained from the relation $\chi+z=2$, which is exact, or from the expression $2\chi=4-d-(\gamma_D+\gamma_\nu)u^*$ [28]. The result is

$$\chi=(4-d)/5. \tag{15}$$

The β exponent $\beta=\chi/z$ is obtained as

$$\beta=(4-d)/(6+d). \tag{16}$$

The exponents given by Eqs. (14)–(16) are obtained to order $\varepsilon=4-d$. In $d=1$ Eq. (14) gives the value 1.4, instead of the exact result $z=3/2$. The latter is due to the fluctuation-dissipation theorem [29], which is valid only in $d=1$. From this we expect that the exponents computed to order ε do not necessarily give the exact value in $d=1$. The situation is to some extent similar to the problem of the anomalous diffusion of a Brownian particle in disordered media. The ε expansion [30] does not give in $d=1$ the exact Sinai's [31] result. The roughness of the directed polymer, $\zeta=1/z$, gives in $d=2$ according to Eq. (14) the value 5/8, which agrees well with existing predictions [1] and [4], Sec.6.4. The value of the β exponent in $d=2$ ($\beta=1/4$), which follows from Eq. (16), is close to the value $\beta=0.24$ obtained in simulations [32–37]. Taking into account that Eq. (16) is only the ε result, the agreement with simulations has to be considered as very good. In $d=3$ Eq. (16) gives the value 1/9, which is smaller than the values obtained in numerical simulations [33,35], and that given by the analytical formula of Kim-Kosterlitz [38]. We do not have an explanation for this discrepancy. Notice that simulations [32,33] give nonideal exponents also for dimensions $d>4$, so that there is a controversy between analytical approaches and simulations. The exponents (14)–(16) differ from those computed in our previous work [39], which in context of the present paper have to be considered as exponents computed to the zero-loop order of RG.

To conclude, it should be emphasized that the present approach is a heuristic one. Although we believe that the arguments stated in this paper give a strong support of the approach, they cannot of course replace a rigorous consideration, which is outstanding. In addition, we want to emphasize the following details of the approach, which give a posteriori legitimization of the latter: (i) the conversion of the dependence of the perturbation expansions on the microscopic cutoff l_0 into the crossover length l_c and consequently into the critical value λ_c is accompanied by changing the pole of the effective coupling constant to a quadratic dependence of the latter on the infrared length; (ii) the shift of the critical dimension, $d\rightarrow d_{uc}=4$, due to (i), matches with $1/(4-d)$ poles in the perturbation expansions, which appear to be responsible for the nonideal values of the strong-coupling exponents.

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